STRATEGIES TO SOLVE A 4x4x3 DOMINEERING GAME

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ABSTRACT

3D Domineering is a three-player variant of the classic two-player combinatorial game, Domineering. Researchers have so far solved a x b x c Domineering games where $a + b + c < 10$ and $a, b, c \geq 2$. In this paper, we solve a 4x4x3 Domineering game. In fact, we show that it is not possible for any of the three players to have a winning strategy.

INTRODUCTION

Domineering

Domineering is a two-player combinatorial game that is played on a grid board of any size (ex. 3x3, 4x6, 10x4, etc.), although it is usually square to be fair to both players. Each player can place a domino piece that covers two adjacent spaces on the board. Both players take turns placing a domino on the board (the domino cannot overlap other domino pieces) until one player is unable to place any more pieces, which results in a loss for that player. One player, referred to as left player or L, can only place pieces vertically on the board. The other player, referred to as right player or R, can only place pieces horizontally on the board. Figure 1 shows a brief example of a Domineering game.

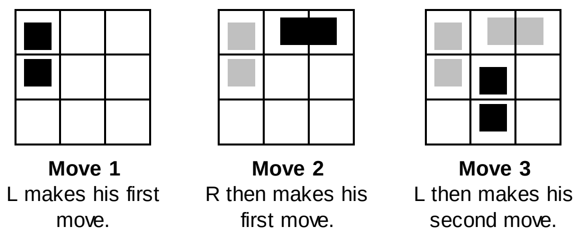

Figure 1: Example of a Domineering Game

3D Domineering

3D Domineering is a three-player variant of the Domineering game. It is played on a rectangular solid board similar to a Rubik's cube whose edges are parallel to the x, y, and z axes. The three players have pieces that occupy two adjacent cubes of the board and take turns in a cyclical order placing them in the 3D board. One player is limited to placing pieces along the x-axis of the board, another player is limited to placing pieces along the y-axis of the board, and the remaining player is limited to placing pieces along the z-axis of the board. If a player is unable to place any more pieces in the board at his turn, then that player is eliminated from the game. The remaining two players will continue to play until one is unable to place any more pieces in the board, resulting in a loss for that player.

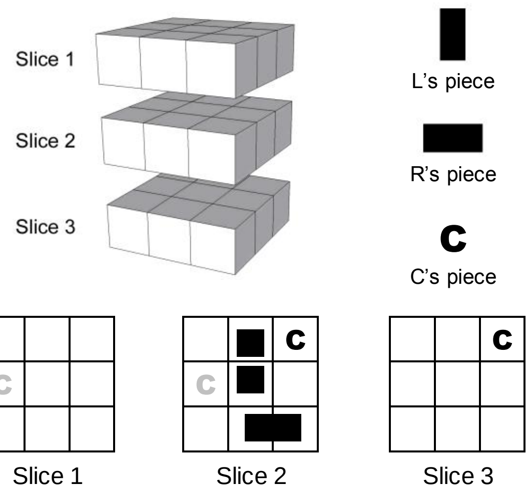

Figure 2: Example of a 3x3x3 Domineering Game

Figure 2 shows an example of a 3x3x3 Domineering game. We marked the three players as L (left), R (right), and C (center). Let's suppose that the turn order for this game is C-L-R. C plays a piece that traverses Slices 1 & 2 on the board. Then L and R play their pieces on Slice 2. C follows up by playing a piece that traverses Slices 2 & 3. All three players continue to take turns until two of the players are no longer able to place any more pieces.

Solving a Domineering Game

A Domineering game is considered *solved* if one can find which player has a winning strategy for each turn order. A winning strategy is a set of moves that will always guarantee a win for one player regardless of the other player's actions (this is also known as *forcing a win*).

Brueker, Uiterwijk and van den Herik created the program DOMI to solve m x n boards where $2 \leq m \leq 8$ and $m \leq n \leq 9$ (Brueker et al. 2000). Bullock later improved on their research by developing a search application called Obsequi that could not only solve those games faster, but also solve a 10 x 10 board, which is currently the largest solved Domineering game (Bullock 2002).

The addition of a third player in 3D Domineering makes analysis of those games more complex than 2D Domineering. This is because you need to consider all possible moves made by three players instead of two, and there is a chance that no player can force a win. Straffin introduced a concept called "decision by player", where one player who is unable to win a three-player game can affect which of the other two players can win with his next move (Straffin 1985). Thus, it is possible that one player cannot force a win no matter what he does if one of his opponents plays a move that causes the other opponent to win. A 3D Domineering game is therefore considered solved when, for each turn order, one can either identify the winning strategy for a player or prove that no player has a winning strategy.

Alessandro Cincotti used an extensive search algorithm to solve 3D (a x b x c) Domineering games where $a + b + c < 10$ and a, b, $c \geq 2$ (Cincotti 2007). Attempts to analyze games where $a + b + c \geq 10$ is difficult to complete because exhaustively searching all of the game's possible moves is computationally expensive.

We address that issue by formulating strategies that can significantly reduce the number of moves to search in a 3D Domineering game. Specifically, we found strategies for two players that can prevent the third player from winning, regardless of the turn order and the third player's actions. Selecting specific moves for the two players, instead of considering all their possible moves, makes it faster to search all possible moves and determine if any of the three players has a winning strategy for each turn order.

We will demonstrate how these strategies prove that a 3x3x3 Domineering game is a *Q* game, a game where no player can force a win in any turn order. We will then expand on those strategies to show that a 4x4x3 game is also a *Q* game.

SOLVING A 3x3x3 DOMINEERING GAME

Alessandro Cincotti had already solved the 3x3x3 Domineering game to be a *Q* game, but we will still present our strategies because they will serve as the foundation to solve a 4x4x3 game.

Proposition: *In a 3x3x3 game of Domineering, it is impossible for one player to force a win if the other two players form an alliance and collude to stop him.*

Since a 3x3x3 game is symmetrical on all sides, if C cannot force a win if L and R collude to stop C, then it also means that L cannot force a win if R and C collude to stop L, and R cannot force a win if L and C collude to stop R. Thus, demonstrating the above proposition will show that no player can force a win in a 3x3x3 Domineering game.

Let's say that L and R form an alliance against C. There exists a two-part strategy that L and R can follow that will prevent C from winning regardless of whether C plays first, second, or third. To be clear, a win for C in a three-player Domineering game means that C still has at least one legal move left after both L and R have no more legal moves.

Notice in Figure 2 that when C places a piece, it traverses either Slices 1 and 2 or Slices 2 and 3. Thus, if all the spaces in Slice 2 are occupied, then C cannot make any more moves. Therefore, the first part of the blocking strategy that L and R should adopt to prevent C from winning is as follows:

Strategy 1: *L & R should place as many pieces as possible on Slice 2 first before placing any pieces on Slices 1 or 3.*

Since L and R are going to be working together to stop C, we should consider this scenario as a special two-player game between C and the L & R alliance. This means that if C is unable to play any legal moves before both L and R are eliminated, then the alliance wins. Furthermore, the alliance has an advantage over C where the alliance can continue to play even if one of its members is unable to place any more pieces.

We will demonstrate an example of the L & R alliance's strategy to cover as much of Slice 2 as possible using the following L-R-C game.

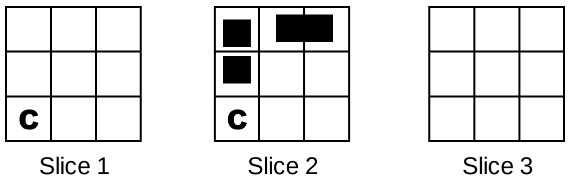

Figure 3: A sample 3x3x3 L-R-C game after one turn

At the start of a 3x3x3 Domineering game, there are nine available spaces in Slice 2. After one complete turn where all three players place their first move in Figure 3, there are four spaces left in Slice 2. One can see in Figure 3 that only one alliance member can move on Slice 2 in his next turn. The other alliance member must move on either Slice 1 or Slice 3, as demonstrated in Figure 4.

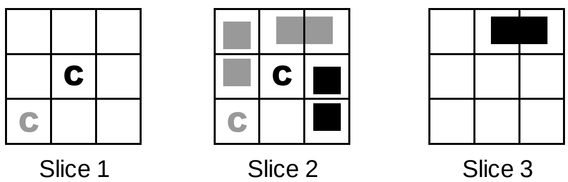

Figure 4: A sample 3x3x3 L-R-C game after two turns

After two complete turns, there is one space left in Slice 2 of Figure 4 (meaning that C has one more move left), but neither alliance member can place any more pieces in Slice

2. However, the alliance can still block C with the second part of its blocking strategy.

Strategy 2: *When there are no more available moves in Slice 2 for the alliance, L and R can cooperatively block C's remaining moves by placing pieces above and below C's available spaces on Slice 2.*

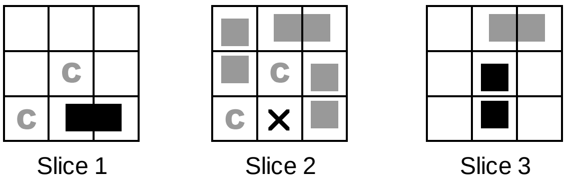

Figure 5: A sample 3x3x3 L-R-C game after three turns

In Figure 5, the alliance's third moves block C from playing on the last available space on Slice 2 (indicated by the x). C has no more available moves, which results in a loss for C.

While this only covers one specific 3x3x3 game, we found that Strategies 1 and 2 are effective in all possible 3x3x3 games (Hurtado 2010). For example, in games where C moves last (L-R-C and R-L-C), the alliance can limit C to a maximum of one space in Slice 2 after two complete turns.

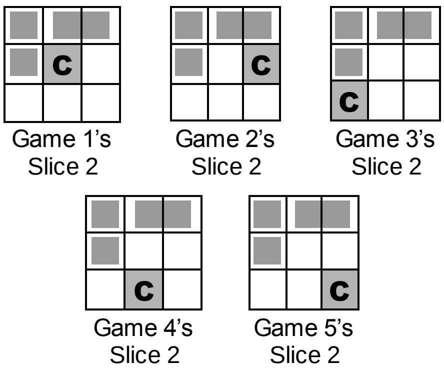

Figure 6: Potential Slice 2s of L-R-C & R-L-C 3x3x3 Games after one complete turn

After L and R make their first moves, there are five possible spaces that C can occupy in Slice 2 during his first turn, as shown by C's shaded moves in the five games of Figure 6. In the upper-left diagram, both alliance members can move in Slice 2 in their next turn, which eliminates C. In the other diagrams, only one alliance member can move in the remaining spaces of Slice 2 in his next turn, which leaves two spaces left. When C occupies one of those spaces with his second move, there is just one space left in Slice 2, which the alliance can cover using Strategy 2 in their next turn, resulting in a loss for C.

In games where C goes second (L-C-R and R-C-L), the alliance can limit C to one space on Slice 2 after C's second turn.

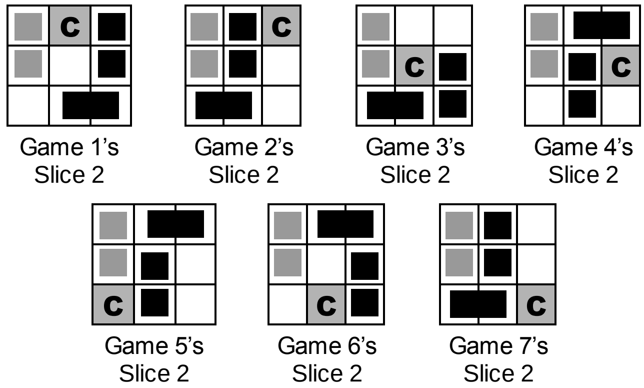

Figure 7: Potential Slice 2s of L-C-R (& R-C-L) 3x3x3 Games before C's second move

After L makes his first move on the upper-left corner of Slice 2, C can occupy seven spaces on Slice 2 in his first move, as shown by C's shaded moves in the seven games of Figure 7. In each game of Figure 7, Slice 2 is left with two spaces when L & R move after C's first turn. When C makes his second move in each game, Slice 2 is left with one space, which the alliance can cooperatively block using Strategy 2 in its next turn, eliminating C from the game. If you rotated all the games in Figure 7 90°, then the examples in Figure 7 would also cover games where R moves first.

For games where C goes first (C-L-R and C-R-L), the alliance can once again limit C to one space on Slice 2 after two turns unless C plays the right moves. We found that C's best opening move is on a space that is neither corner nor center. This is because it allows him to prevent the alliance from covering any more spaces on Slice 2 with a second move demonstrated in Figure 8.

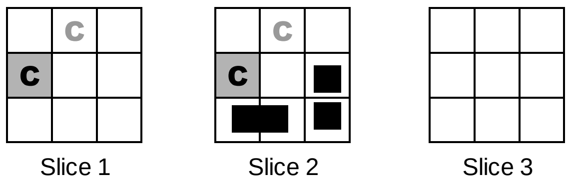

Figure 8: C prevents the alliance from blocking Slice 2 and reserves a move

As long as C plays the moves indicated in Figure 8, he will have three available moves in Slice 2 after his second turn instead of one. Furthermore, placing both his moves on Slices 1 & 2 allows C to protect a move in the upper-left corner of the board from any blocking attempts by L or R. This tactic is referred to as *reserving a space*.

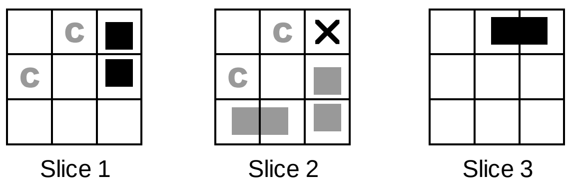

Figure 9: L & R block one of C's moves

Since the alliance is unable to play on Slice 2, they must perform Strategy 2 and cooperatively block the upper-right space of the board, as seen in Figure 9 (C's blocked move is indicated by the x). This leaves C with just two moves left after two complete turns. C could play on his reserved space (upper-left corner) in his next turn, but this allows the alliance to cooperatively block the center space in their next turn, eliminating C. Therefore, C's better move is to play on the center space.

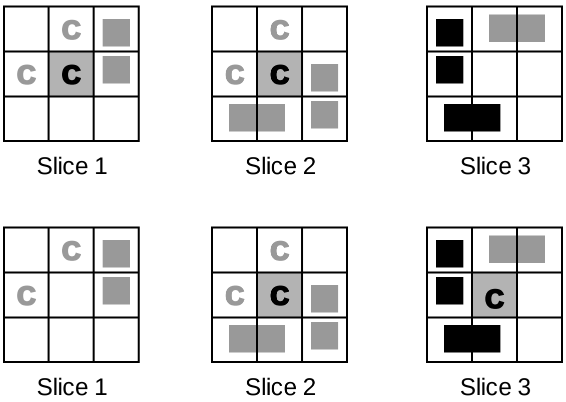

Figure 10: Both games show C playing on the center space

Figure 10 shows the two moves C could play that cover the board's center space (which are highlighted in gray). The alliance is unable to cooperatively block C's last move since it is reserved, so it instead plays on Slice 3 in both games. After C plays his last move on the upper-left corner, one can see in the two games of Figure 10 that both L and R have still one more move left in the game. Thus, C loses because he is eliminated before both L and R.

Regardless of turn order and where C plays, the alliance is always able to prevent C from winning as long as it follows the winning strategy. Thus, we have reconfirmed what Cincotti already solved (a 3x3x3 game is a *Q* game) using Strategies 1 & 2. In the next section, we expand on those strategies to solve a 4x4x3 Domineering game.

SOLVING A 4x4x3 DOMINEERING GAME – PART 1

A 4x4x3 game is not symmetric on all sides, so this affects our analysis of where L, R, and C play in that game's board. We previously mentioned that the edges of a 3D Domineering board are parallel to the x-, y-, and z-axes. In this paper, we define that R, L, and C play along the x-, y-, and z-axes respectively.

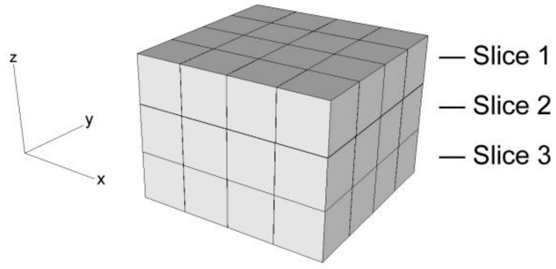

Figure 11: Establishing the x-, y-, and z- axes of a 4x4x3 game

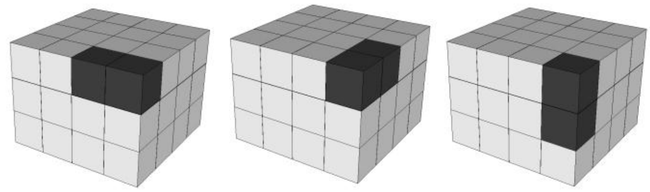

Figure 12: R's, L's, and C's moves (highlighted in dark gray) are respectively parallel to the board's x-, y-, and z-axes

We establish these definitions to make it clear where R, L, and C are playing when presenting how to solve a 4x4x3 game (in an *a* x *b* x *c* game, R plays on *a*, L plays on *b*, and C plays on *c*).

We will first show that the alliance can prevent C from winning a 4x4x3 Domineering game if C plays on the 3 component. We will then show that the alliance can prevent C from winning a 4x4x3 Domineering game if C plays on the second 4 component (which will be subsequently referred to as a 4x3x4 game).

Demonstrating that C cannot force a win in a 4x4x3 game if L and R collude against him, regardless of turn order or whether C plays on the 3 component or the second 4 component, proves that no player has a winning strategy in a 4x4x3 game. This is because if C cannot force a win if L and R team up against C, it also means that L cannot force a win if R and C team up against L, and R cannot force a win if L and C team up against R.

Proposition: C *is unable to force a win in a 4x4x3 game, where C plays on the 3 component, if the L & R alliance teams up against him.*

The strategies that L & R can use to do this are similar to the strategies that prevented C from winning a 3x3x3 game.

Strategy 1: *L & R should place as many pieces as possible on Slice 2 first before placing any pieces on Slices 1 or 3.*

Strategy 2: *When there are no more available moves in Slice 2 for the alliance, L and R can cooperatively block C's remaining moves by placing pieces above and below C's available spaces on Slice 2.*

After analyzing all possible moves that C can do in each turn order, we have found that the alliance can reduce the number of C's available spaces in Slice 2 to a maximum of three spaces within three turns when following Strategy 1 (Hurtado 2010). In games where C goes last, the alliance can limit C to one space on Slice 2 within three turns. In games where C goes first or second, on the other hand, the alliance can only limit C to three spaces on Slice 2 within three turns. In this section, we'll focus on the latter games.

Figure 13 shows an example of a 4x4x3 game where C has three moves left after three turns.

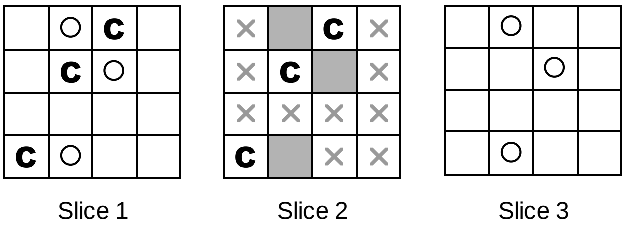

Figure 13: A potential 4x4x3 game with C having three spaces left (indicated by the shaded squares and Os)

The shaded squares in Slice 2 represent C's available spaces on that slice. The Os in Slices 1 and 3 represent where the other half of C's move could go when he plays on an available space in Slice 2. The gray x's in Slice 2 represent potential pieces places by L and R when executing Strategy 1. Although this diagram represents a specific 4x4x3 game, we claim that no matter how you configure C's three moves and C's three available spaces in Slice 2 in a 4x4x3 game, the alliance is still able to cooperatively block at least one of C's available moves using Strategy 2 and successfully prevent C from winning.

Here's how the alliance is able to block at least one of C's available moves. There exists no configuration on a 4x4 slice where C's three pieces will protect all three of his potential moves. This is because in order for C to protect one of his moves from the alliance on either Slice 1 or Slice 3, he needs at least two pieces.

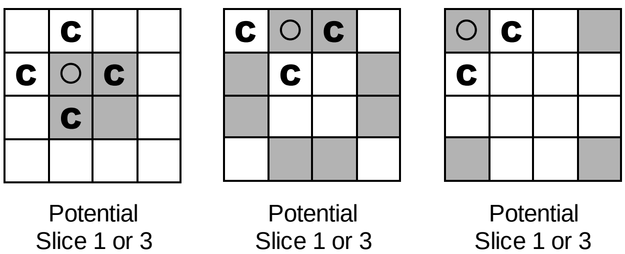

Figure 14: A demonstration of how C can protect his potential moves (Os)

Figure 14 shows how many C pieces are needed to protect a C potential move (O) depending on its location. The first diagram of Figure 14 demonstrates that four C pieces are needed to protect potential C moves in center spaces (which are shaded). This scenario, however, is not possible because C has only moved three times before this scenario occurred (there would only be three C pieces, not four).

Figure 14's second diagram reveals that three C pieces are needed to protect Os that are on spaces that are neither corner nor center (which are shaded). Since all of C's three pieces are protecting only one potential space, this leaves the other potential moves unprotected, which can be covered by an alliance member.

Figure 14's third diagram is the best case for C because it shows that only two C pieces are needed to protect Os that are on corner spaces (which are shaded). However, if we were to place another O on that board, C could not protect it with his third piece because we just showed that C needs at least two pieces to protect a potential move.

If C cannot protect two potential moves with his three pieces, he certainly cannot protect three potential moves with his three pieces. Therefore, we can confidently state that the alliance can cooperatively block at least one of C's potential moves

Why is this important? We stated that C has three moves left after three turns. If C occupies one of them in his next turn, and the alliance cooperatively block one of C's remaining moves in its next turn, then C will only have one move left by the fifth turn. There would still be plenty of spaces in Slices 1 and 3 for the alliance to move even after C plays on his last available space, which is a loss for C. In games where C goes last, it's worse for C because C only has one available space in Slice 2 after three turns, which the alliance can cooperatively block in their next turn.

Therefore, we have shown that for all possible games on a 4x4x3 board, where C plays on the 3 component, C cannot force a win if the alliance teams up against him and uses Strategies 1 & 2. However, we have not completely solved the 4x4x3 game yet. We still need to show that the alliance can prevent C from winning even if you rotate the 4x4x3 board so that C plays on the second 4 component (4x3x4).

SOLVING A 4x4x3 DOMINEERING GAME – PART 2

To complete our demonstration that a 4x4x3 game is solved to be a *Q* game, we must demonstrate the following:

Proposition: *C is unable to force a win in a 4x3x4 game, where C plays on the 4 component, if the L & R alliance teams up against him.*

Similar to the 3x3x3 and 4x4x3 (where C plays on the 3 component) games, the alliance has a two-part winning strategy to prevent C from winning. Since C is able to move on four slices in a 4x3x4 game instead of three, the alliance will need to cover two slices of the game board in order to prevent C from making any more moves. Thus, the first part of the alliance's winning strategy in 4x3x4 games is:

Strategy 1: *Cover as much of Slices 1 & 3 as possible for the duration of the game until L and R are unable to move on either slice.*

It is crucial that the alliance commits to playing on either Slices 1 or 3 until they cannot move on those slices anymore. Deciding which of the two slices to play will depend on where C last played.

C will play on Slices *j* and *j+1* in his turns, where *j* = 1, 2, or 3. If *j* = 1 or 3, the alliance should play on the same Slice *j*. If *j* = 2, then the alliance should play on Slice 3. However, if an alliance member is unable to play on one of the slices in the Slices 1 & 3 pair, then he should move on the other slice in the pair if a move is available. For example, let's say C played on Slices 1 and 2. According to our strategy, the alliance should move on Slice 1 on its next turn. If L or R is unable to move on Slice 1, then that alliance member should play on Slice 3 if a move is available. If an alliance member is unable to play on either Slice 1 or Slice 3, then he must follow the second part of the alliance's winning strategy.

Strategy 2: *The alliance must cover the corresponding squares above and below any remaining moves for C to block him from playing them. If the alliance is unable to cooperatively block an available move for C, then both players should strive to either block available C moves that only need one piece to block or partially block an available C move.*

There are two cases to consider when the alliance starts the second part of its winning strategy. The first case is when both alliance members are unable to move on Slices 1 and 3 in their next turn. The second case is when only one alliance member is unable to move on Slices 1 and 3 in his next turn. We will start the analysis of Strategy 2 with the first case.

Case 1: *Both alliance members can no longer play on Slices 1 and 3*

If both alliance members can no longer move on Slices 1 and 3 in their next turn, then they should cooperatively block an available C space in their next turn if the opportunity exists. The following diagram shows an example of this strategy.

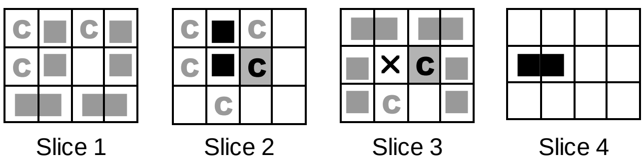

Figure 15: L & R cooperatively block a C potential move

In this sample 4x3x4 game, the alliance plays moves above and below an available C move to block it (indicated by the x) after C's fifth turn (indicated by the shaded square).

However, the alliance needs to be aware which of C's potential moves they cooperatively block. The following diagrams demonstrate why.

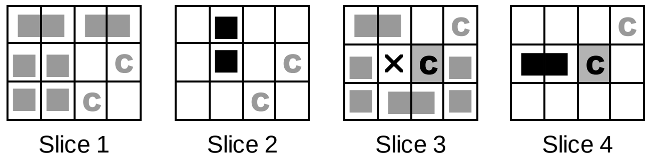

Figure 16: L & R cooperatively block one C move

In Figure 16's sample 4x3x4 game, the moves L & R made after C's fourth move (indicated by the shaded square) are not an optimal winning strategy for the alliance. This is because there is another pair of moves that the alliance can make that can block two of C's available moves (indicated by the two x's in Figure 17).

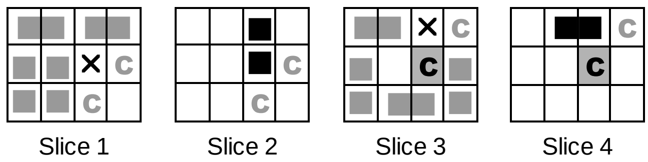

Figure 17: L & R cooperatively block two C moves

Thus, the following summarizes the alliance's optimal winning strategy for Case 1.

- *If there are cooperative moves that can block more than one of C's available moves, then the alliance should play them.*
- *If that is not possible, then the alliance should play a pair of cooperative moves in their next turns that can block one of C's available moves.*
- *If cooperative blocking moves are not available, then both players should block different available moves for C in their next turns.*
- *If none of the above applies to an alliance member, then the member will need to pick a valid move. A valid move in this case is one that would not prevent the other alliance member from playing a piece that would have blocked one of C's available moves.*

We continue the analysis of Strategy 2 with the second case.

Case 2: *Only one alliance member can no longer play on Slices 1 and 3*

The one alliance member who cannot play on Slice 1 or Slice 3 should look for moves that can block one of C's available spaces using one piece. The following diagram shows an example of this strategy.

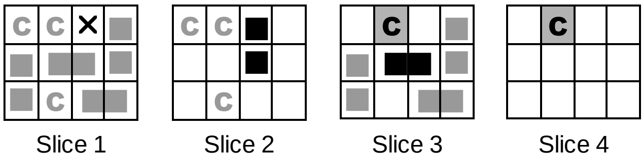

Figure 18: A sample 4x3x4 game (L-C-R)

After C's fourth turn (indicated by the shaded square) in Figure 18's sample game, R plays on Slice 3. L is unable to play on Slice 1 or Slice 3, so he places a move on Slice 2 that blocks C (indicated by the x).

This alliance member, however, must be careful if he happens to go before the other alliance member who can play on Slice 1 or 3. Here's why.

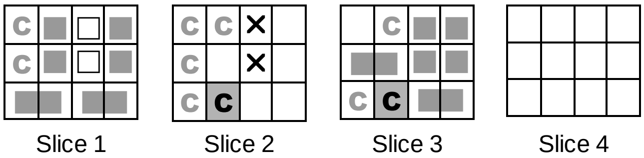

Figure 19: A sample 4x3x4 game (L-C-R)

It is R's turn after C's fifth turn in Figure 19's sample game and he can neither play on Slices 1 or 3. L still has a move left on Slice 1 (indicated by the open squares) that would block two of C's spaces in Slice 2 (indicated by the x's). If R were to make a move that would overlap one of those x's, it would be a wasted move for the alliance since R could have blocked another C move that would not have been blocked by L's move, as indicated in Figure 20.

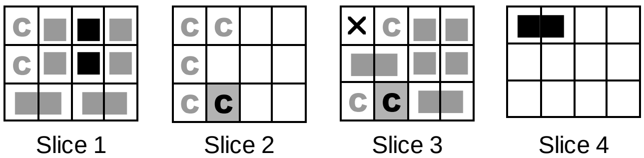

Figure 20: C is eliminated in this game

The following summarizes the alliance's winning strategy for Case 2.

If only one alliance member cannot play on Slice 1 or Slice 3 in his next turn, he should play a move that can fully block an available C space using only one piece. If that member goes first, however, he must not block an available C space that will also be blocked by the other alliance member's next move on a slice that is in the Slices 1 & 3 pair.

We developed an application that simulated all possible 4x3x4 games that followed the strategies described in this section for two of the players (in this case, L & R) and considered all possible moves made by the third player (C). The application generated six text files from the 4x3x4 game simulation, one for each turn order (C-L-R, C-R-L, L-R-C, R-L-C, L-C-R, R-C-L). The following is an excerpt from one of those text files:

```
Program started at: 1284406810
Dimensions of Domineering board: 4x3x4
Turn order for all games in this simulation: left, right, center

Game Number: 1 + (10000000 x 0)
  Winner: left/right
    left:    0,1,0  3,1,0  2,1,0  3,0,2  1,1,2  2,1,1
    right:   0,0,0  2,0,0  0,0,2  2,2,2  1,0,3  2,1,3
    center:  1,1,0  1,2,0  0,1,1  0,2,1  2,0,1
    # of Remaining Moves: - left: 5 - right: 5 - center: 0

...

Game Number: 114377 + (10000000 x 0)
  Winner: left/right
    left:    0,1,0  0,1,2  1,1,2  3,1,0  1,1,0  0,0,1
    right:   0,0,0  0,0,2  2,0,2  2,0,0  2,1,1  0,2,1
    center:  3,2,2  3,1,2  2,2,2  2,1,2  2,2,0
    # of Remaining Moves: - left: 5 - right: 7 - center: 0

Program finished at: 1284406831
```

Figure 21: An excerpt from the output of the 4x3x4 L-R-C game simulation

The first lines of the output indicate the program's start time, the dimensions of the board, and the turn order for all games simulated by the application. The simulation stops analyzing a game when either C wins or has no more available moves (regardless of whose turn it is). It outputs how many times the simulation stopped analyzing a game (indicated in the Game Number line), whether C won or not, and what moves each player made for that specific game (indicated by a three-number coordinate that marks the first half of the player's move). It also writes the number of remaining moves left for each player. When the application finishes traversing through all possible games, it writes the time before it quits.

In each turn order output file, performing a search for "Winner: center" yielded no results. This is how we determined that *there are no games where C won when the alliance performed its winning strategy*. If you were to take a random game from any of the output files and played the moves indicated in that game, it will show that the alliance performed their winning strategy to prevent C from winning.

Table 1 outlines how long the application took to simulate all possible games for each turn order and how many games it simulated.

Turn Order	Duration	Number of games
CLR	251 seconds	1,415,738
CRL	259 seconds	1,455,637
LCR	81 seconds	453,729
RCL	83 seconds	458,325
LRC	21 seconds	114,377
RLC	21 seconds	116,839
Total:	**716 seconds**	**4,014,645**

Table 1: 4x3x4 3D Domineering simulations using two-part winning strategy

We also modified the application to simulate all possible 3x3x3 C-L-R games (where none of the players use any type of strategy) to compare results with the 4x3x4 simulations in Table 1.

Turn Order	Duration	Number of Games
3x3x3 – CLR	Approx. 8 hours	1,034,224,512

Table 2: 3x3x3 3D Domineering simulations

You may have noticed that we simulated 3x3x3 games in Table 2 instead of 4x3x4 games. This is because simulating all possible 4x3x4 games would have taken the application an extremely long time to finish, as there are a substantial number of possible moves to traverse.

Furthermore, the simulations in Table 2 are not actually complete simulations—they only considered all possible moves made by the player whose turn was next at each position. This differs from a complete analysis of a 3D Domineering game because one must consider all possible moves that can be made by *all three players* at each game's position, regardless of whose turn it is. Even with a 3x3x3 board, this would have created a very large number of moves for the application to traverse, which is why we limited the simulations to only consider moves for one player at each position. Nonetheless, we were able to estimate the total number of moves of a completely analyzed 3x3x3 game using the Number of Games from Table 2, which we found to be approximately 201,280,264,483 (Hurtado 2010).

We can make the case that the total number of possible games for a 4x3x4 game would be significantly higher than the number of games for a 3x3x3 game. We make this point because the total time and number of games from our 4x3x4 simulations would clearly be significantly less than a complete analysis of a 4x3x4 game.

CONCLUSION

We have demonstrated in the last two sections that C cannot force a win in a 4x4x3 game if L and R collude against him, regardless of turn order or whether C plays on the 3 component or the second 4 component. Therefore, we have shown that the 4x4x3 is solved to be a *Q* game, as no player can force a win in a 4x4x3 game, regardless of board orientation or turn order.

FUTURE WORK

Solving a 4x4x3 Domineering game using the strategies outlined in the previous sections opens the possibility of solving 3D Domineering boards of larger sizes without having to analyze all possible moves for those games. We previously determined that the alliance's optimal strategy to prevent C from winning 3x3x3 and 4x4x3 (where C plays on the 3 component) games is to cover as much of the middle slice as possible. One consideration for future work is to investigate if there is a limit to how large a game of m x n x 3 (where $m > 4$, $n > 4$, and C plays on the 3 component) can be where blocking the middle slice remains an effective strategy for the alliance to block C.

Another consideration for future work is to test if having the alliance play on alternating slices is an effective strategy to prevent C from winning regardless of how many slices C can play on. We found that when C plays on four slices in a 4x3x4 game, the strategy to block C is to play on alternating slices (Slices 1 & 3). If C played on five slices (for example, 5x5x5), can the alliance prevent C from winning by playing on Slices 2 & 4? And if C played on six slices (for example, 6x6x6), can the alliance prevent C from winning if they played on Slices 1, 3, and 5? What if the board size was skewed to C's favor, such as a 3x3x8 board, with C playing on the 8 component? Would playing on alternate slices still be enough for the alliance to prevent C from winning? How big does the board have to skew in C's favor before C can force a win despite the other two players teaming up against him? These are questions that can be explored with further research.

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BIOGRAPHY

JONATHAN HURTADO was born in Cali, Colombia and obtained his undergraduate degree in Computer Science from New York University in 2001. After working as a web developer for several years, he pursued a graduate degree in Computer Science from the Digipen Institute of Technology, which he obtained in 2010. He is currently working in Digipen's Singapore campus.